

Efficient k -Party Voting with Two Choices

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Abstract

We consider the problem of distributed k -party voting with two choices as well as a simple modification of this protocol in complete graphs. In the standard version, we are given a graph in which every node possesses one of k different opinions at the beginning. In each step, every node chooses two neighbors uniformly at random. If the opinions of the two neighbors coincide, then this opinion is adopted. It is known that if $k = 2$ and the difference between the two opinions is $\Omega(\sqrt{n \log n})$, then after $O(\log n)$ steps, every node will possess the largest initial opinion, with high probability.

We show that if $k = O(n^\epsilon)$ for some small ϵ , then this protocol converges to the initial majority within $O(k \cdot \log n)$ steps, with high probability, as long as the initial difference between the largest and second largest opinion is $\Omega(\sqrt{n \log n})$. Furthermore, there exist initial configurations where the $\Theta(k)$ bound on the run time is matched. If the initial difference is $O(\sqrt{n})$, then the largest opinion may lose the vote with constant probability. To speed up our process, we consider the following variant of the two-choices protocol. The process is divided into several phases, and in the first step of a phase every node applies the two choices protocol. If a new opinion is adopted, the node remembers it by setting a certain bit to `TRUE`. In the subsequent steps of that phase, each node samples one neighbor, and if the bit of this neighbor is set to `TRUE`, then the node takes the opinion of this neighbor and sets its bit to `TRUE` as well. At the end of the phase, the bits are reset to `FALSE`. Then, the phases are repeated several times. We show that this modified protocol improves significantly over the standard two-choices protocol.

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1 Introduction

Distributed voting is a fundamental problem in distributed computing. In the original problem, we are given a network of players modeled as a graph $G = (V, E)$ with $|V| = n$ and $|E| = m$. Each player in the network starts with one initial opinion (we sometimes call color) from a finite set of possible opinions. Then the voting process runs either synchronously in discrete rounds or asynchronously according to some (possibly random) clock. During these rounds in the synchronous case, or upon activation in the asynchronous case, the players are allowed to communicate with their direct neighbors in the network with the main goal to eventually agree on one of the initial opinions. If all nodes agree on one opinion, we say this opinion *wins* and the process *converges*. Typically, one would demand from such a voting procedure to run accurately, that is, the opinion with the highest number of initial supporters should win with decent probability, and efficiently, that is, the voting process should converge within as few communication steps as possible. Additionally, voting algorithms are usually required to be simple, fault-tolerant, and easy to implement [31, 33].

Distributed voting algorithms have applications in various fields. In distributed computing these contain for instance consensus [31] and leader election [13]. Early results in other areas can be attributed to distributed databases [29] where voting algorithms have been used to serialize read and write operations. In game theory, distributed voting can be used to analyze social behavior [22]. Also, the existence and characterization of so-called *monopolies*, which are sets of nodes that dominate the outcome of the voting process, have been investigated [44, 12, 45, 7]. Various processes based on the *majority rule* have been analyzed in the study of *influence networks* [39, 28], and they have been used to measure the competition of opinions in social networks [42]. Variants of these processes have been used for distributed community detection [47, 36, 20]. In computational sciences, voting processes can be used to model chemical reaction networks [25], neural and automata networks [30], and cells' behavior in biology [15, 14].

Population Protocols The first major line of work on majority voting considers *population protocols*, in which the nodes usually act asynchronously. In its basic variant, nodes are modeled as finite state machines with a small state space. Communication partners are chosen either adversarially or randomly, see [6, 5] for a more detailed description. Angluin et al. [4] propose a 3-state (i.e., constant memory) population protocol for majority voting (i.e., $k = 2$) on the clique to model the mixing behavior of molecules. We refer to their communication model as the *sequential model*: each time step, an edge is chosen uniformly at random, such that only one pair of nodes communicates. If the initial bias $c_1 - c_2$ is $\omega(\sqrt{n} \log n)$, their protocol lets all nodes agree with high probability on the majority opinion in $O(n \cdot \log n)$ steps. Mertzios et al. [41] show that this 3-state protocol fails on general graphs, i.e., there are infinitely many graphs on which it returns the minority opinion or has exponential run time. They also provide a 4-state protocol for *exact* majority voting, which *always* returns the majority opinion independently of the initial bias in time $O(n^6)$ on arbitrary graphs and in time $O(\log n / (c_1 - c_2) \cdot n^3)$ on the clique. This result is optimal in that no population protocol for exact majority can have fewer than four states. In a recent paper, Alistarh et al. [3] give a sophisticated protocol for $k = 2$ on the clique in the sequential model. It solves exact majority and has with high probability *parallel run time* $O(n \log^2 n / (s \cdot (c_1 - c_2)) + \log^2 n \cdot \log s)$, where s is the number of states that must be in the range $s = O(n)$ and $s = \Omega(\log n \cdot \log \log n)$. Note that parallel run time in the sequential model is the number of sequential time steps divided by n [3].

Pull Voting The second major research line on k -party voting, which is also known as plurality consensus, has its roots in gossiping and rumor spreading. Communication in these models is often restricted to pull requests, where nodes can query other nodes' opinions and use a simple rule to update their own opinion. Note that the 3-state protocol from [4] fits into this model. See [44] for a slightly dated but thorough survey.

One straightforward variant is the so-called *pull voting* running in discrete rounds, during which each player contacts a node chosen uniformly at random from the set of its neighbors and adopts the opinion of that neighbor. The two works by Hassin and Peleg [31] and Nakate et al. [43] have considered the discrete time two-party voter model on connected graphs. In these papers, each node is initially assigned one of two possible opinions. Their main result is that the probability for one opinion \mathcal{A} to win is $P_{\mathcal{A}} = d(\mathcal{A})/(2m)$, where $d(\mathcal{A})$ denotes the sum of the degrees of all vertices supporting opinion \mathcal{A} . It has been furthermore shown by Hassin and Peleg [31] that the expected time for the two-party voting process to converge to one opinion on general graphs can only be bounded by $O(n^3 \log n)$.

In [16], Cooper et al. show that the convergence time until a single message emerges for pull voting on any connected graph $G = (V, E)$ is asymptotically almost always in $O(n/(\nu(1 - \lambda_2)))$. In this bound, λ_2 is the second largest eigenvalue of the transition matrix of a random walk on the graph G . The parameter ν measures the *regularity* of G with $1 \leq \nu \leq n^2/(2m)$, where the equality $\nu = 1$ holds for regular graphs. Tighter bounds for the expected completion time on random d -regular graphs have been shown in [19].

From the result in [31] one can conclude that even if only a single vertex v initially holds opinion \mathcal{A} , this opinion still wins with probability $P_{\mathcal{A}} = \deg(v)/(2m)$. Moreover, the expected convergence time is at least $\Omega(n)$ on many graphs, such as regular expanders and complete graphs. Taking into account that solutions to many other fundamental problems in distributed computing such as information dissemination [34] or aggregate computation [35] are known to run much more efficiently, this leaves room for improvement (see [17]).

To address this issue, [17] considered a modified version of the two party pull voting process. In this modified process called *two-sample voting*, one is given a graph $G = (V, E)$ where each node has one of two possible opinions. The process runs in discrete rounds during which, other than in the classical pull voting, every node is allowed to contact two neighbors chosen uniformly at random. If both neighbors have the same opinion, this opinion is adopted, otherwise the calling vertex retains its current opinion in this round. In their model, for random d -regular graphs, with high probability all nodes agree after $O(\log n)$ steps on the largest initial opinion, provided that $c_1 - c_2 = K \cdot (n\sqrt{1/d + d/n})$ for K large enough. For an arbitrary d -regular graph G , they need $c_1 - c_2 = K \cdot \lambda_2 \cdot n$ [17].

One extension is five-sample voting in d -regular graphs with $d \geq 5$, where in each round at least five distinct neighbors are consulted. Abdullah and Draief showed an $O(\log_d \log_d n)$ bound [1], which is tight for a wider class of voting protocols. A more general analysis of multi-sample voting has been conducted by Cruise and Ganesh [21] on the complete graph. In the more recent work by Cooper et al. [18], the results from [17] have been extended to general expander graphs, cutting out the restrictions on the node degrees but nevertheless proving that the convergence time for the voting procedure remains in $O(\log n)$.

Becchetti et al. [9], as [17], consider a similar update rule on the clique for k opinions. Here, each node pulls the opinion of three random neighbors and adopts the majority opinion among those three (breaking ties uniformly at random). They need $O(\log k)$ memory bits and prove a tight run time of $\Theta(k \cdot \log n)$ for this protocol, given a sufficiently high bias $c_1 - c_2$. In another recent paper, Becchetti et al. [8] build upon the idea of the 3-state population protocol

from [4]. Using a slightly different time and communication model, they generalize the protocol to k opinions. In their model, nodes act in parallel and pull the opinion of a random neighbor each round. If $n_1 \geq (1 + \varepsilon) \cdot n_2$ for a constant $\varepsilon > 0$, if $k = O((n/\log n)^{1/3})$, and if given a memory of $\log k + O(1)$ bits, they agree with high probability on the plurality opinion in time $O(\text{md}(\mathbf{c}) \cdot \log n)$ on the clique. Here, $\text{md}(\mathbf{c})$ is the so-called *monochromatic distance* that depends on the initial opinion distribution \mathbf{c} .

In contrast to all the results above for $k > 2$ opinions, we only require a bias of size $O(\sqrt{n \log n})$.

Further Models Aside from the two research lines mentioned above, there is a multitude of related but quite different models. They differ, for example, in the consensus requirement, the time model, or the graph models. This paragraph gives merely a small overview over such model variants. For details, the reader is referred to the corresponding literature.

In one very common variant of the voter model [32, 24, 31, 16, 38, 2, 37, 40], one is interested in the time it takes for the nodes to agree on *some* (arbitrary) opinion. Notable representatives of this flavor are [23, 10]. Both papers consider a variant, in which the final color can be on an arbitrary opinion, instead of on the largest initial opinion. They have the additional requirement that the agreement is robust even in the presence of adversarial corruptions. Another variant [46] of distributed voting considers the 3-state protocol from [4] for two opinions on the complete graph, but in a continuous time model. A third variant [1] considers majority voting on special graphs given by a degree sequence. Other protocols such as the one presented in [26] guarantee convergence to the majority opinion. The authors of [26] analyze their protocol for 2 opinions. Moreover, Berenbrink et al. [11] use load balancing algorithm to solve the plurality consensus in general graphs and general bias. However, in the setting where the bias is of order $\sqrt{n \log n}$ and the number of colors is polynomial in n , their run time becomes substantial.

1.1 Two-Choices Model

We are given a graph $G = (V, E)$ with $|V| = n$ nodes and $|E| = m$ edges. On this network, we run the following process in discrete time steps $t \in \mathbb{N}^0$, starting with time $t = 0$. Initially, the nodes are partitioned into k groups representing k colors $\mathcal{C}_1, \dots, \mathcal{C}_k$. We will denote the set of all colors as $C = \{\mathcal{C}_1, \dots, \mathcal{C}_k\}$. We will occasionally abuse notation and use \mathcal{C}_i to denote the *set* of all vertices having color \mathcal{C}_i . At time t , every node randomly chooses two neighbors (uniformly, with replacement). If the chosen nodes' colors coincide and this color is not the color of the node itself at time $t - 1$, then the node switches to this new color at time t . We denote this process as the *k-party voting process with two choices*. We will say that the process *converges* when all nodes have the same color.

Let t denote an arbitrary but fixed time step and c_1, \dots, c_k the numbers of nodes of colors $\mathcal{C}_1, \dots, \mathcal{C}_k$ at time step t . W.l.o.g. assume that at every time step t the colors are ordered in descending order such that $c_1 \geq c_2 \geq \dots \geq c_k$. Now let f_{ij} denote the random variable for the *flow* from color \mathcal{C}_i to color \mathcal{C}_j , that is, f_{ij} at a given time step t represents the number of nodes which had color \mathcal{C}_i at the previous time step $t - 1$ and switched to color \mathcal{C}_j at time t . We will use c'_1, \dots, c'_k to denote the number of nodes of corresponding colors after the switching has been performed, that is, at time step $t + 1$.

For simplicity of notation, we will assume that in the following the dominating color \mathcal{C}_1 is denoted as \mathcal{A} with $a = c_1$. Furthermore, we will use \mathcal{B} to denote the second largest color \mathcal{C}_2 of size $b = c_2$. Also, we will use $f_{\mathcal{AB}}$ and $f_{\mathcal{BA}}$ to denote $f_{1,2}$ and $f_{2,1}$, respectively.

1.2 Our Contribution

Our first main contribution is an extension of the results by Cooper et al. [17] on the complete graph to more than two colors. That is, in our model we assume that every node of the K_n initially has one of k possible opinions where $k = O(n^\epsilon)$ for some small positive constant ϵ . In the following, we state this as our first main theorem.

Theorem 1. *Let $G = K_n$ be the complete graph with n nodes. Let c_1 be the size of the largest opinion at the beginning of the process. The k -party voting process with two choices defined in Algorithm 1 on G converges to \mathcal{A} with high probability¹ within $O(n/c_1 \cdot \log n)$ time steps, if $c_1 - c_2 \geq z \cdot \sqrt{n \log n}$ for some constant z .*

As we show later, the required number of time steps is $\Omega(n/c_1 + \log n)$. Moreover, we show that if $c_1 - c_2 = O(\sqrt{n})$, then \mathcal{B} wins with constant probability. Furthermore, we investigate a modified model which we call the *memory model*. This improved model is described in full detail in Section 3. In this model, we allow each node to store and transmit one additional bit. As stated in the following theorem, this allows us to reduce the run time from $O(n/c_1 \cdot \log n)$ to $O((\log(c_1/(c_1 - c_2)) + \log \log n) \cdot (\log k + \log \log n)) = O(\log^2 n)$ while still the dominating color wins with high probability, assuming only a slightly larger initial bias towards the dominating color than in the two-choices approach. The bound becomes $O(\log \log n \cdot (\log k + \log \log n))$ for $c_1 \geq c_2(1 + 1/\log^{O(1)} n)$. If we assume that a tight upper bound on n/a is known to the nodes, the run time of Algorithm 2 can further be improved to $O((\log \log n) \cdot (\log n/a + \log \log n))$. The theorem is formally stated as follows.

Theorem 2. *Let $G = K_n$ be the complete graph with n nodes. The memory voting process defined in Algorithm 2 on G converges within $O((\log(c_1/(c_1 - c_2)) + \log \log n) \cdot (\log k + \log \log n))$ time steps to \mathcal{A} , with high probability, if $c_1 - c_2 \geq z \cdot \sqrt{n \log^3 n}$ for some constant z .*

Note that also in the classical two-choices protocol each node implicitly is assumed to have local memory, which is used, e.g., to store its current opinion. The main difference between the classical model and the memory model is that in the memory model each node also transmits one additional bit along with its opinion when contacted by a neighbor. In contrast to existing work considering $k > 2$, our algorithm ensures that the dominant color \mathcal{A} wins within a small (at most $O(\log^2 n)$) number of rounds even if the bias is only $O(\sqrt{n \log^3 n})$.

2 k -Party Voting with Two Choices

In this section we prove our first main theorem stated in Theorem 1. The algorithm discussed in this section is formally defined in Algorithm 1.

Observe that in the complete graph the number f_{ij} of nodes switching from \mathcal{C}_i to \mathcal{C}_j has a binomial distribution with parameters $f_{ij} \sim B(c_i, c_j^2/n^2)$. Clearly, the expectation and variance of f_{ij} are

$$\mathbb{E}[f_{ij}] = \frac{c_i \cdot c_j^2}{n^2} \quad \text{and} \quad \text{Var}[f_{ij}] = \frac{c_i \cdot c_j^2 (n - c_j) (n + c_j)}{n^4}.$$

Observe that if $a \geq (1/2 + \varepsilon_1)n$ for some constant $\varepsilon_1 > 0$, the process converges within $O(\log n)$ steps with high probability. This follows from [17] since in the case of $a \geq (1/2 + \varepsilon_1)n$

¹Throughout this paper, the expression *with high probability* means a probability of at least $1 - n^{-\Omega(1)}$.

Algorithm two-choices($G = (V, E)$, $\text{color} : V \rightarrow C$)

```

for round  $t = 1$  to  $|C| \cdot \log |V|$  do
  at each node  $v$  do in parallel
    let  $u_1, u_2 \in N(v)$  uniformly at random;
    if  $\text{color}(u_1) = \text{color}(u_2)$  then
       $\text{color}(v) \leftarrow \text{color}(u_1)$ ;

```

Algorithm 1: Distributed Voting Protocol with Two Choices

the process is stochastically dominated by the two color voting process. In order to increase readability we assume in the following that $a \leq n/2$. Furthermore, observe that $a > n/k$, since \mathcal{A} is the largest of k color classes. We start with the following definitions.

Let $S \subseteq C$ be a set of colors. We will use the random variables f_{Si} and f_{iS} to denote the sum of all flows from color \mathcal{C}_i to any color in S , which gives us in expectation

$$\mathbb{E}[f_{Si}] = \sum_{\mathcal{C}_j \in S} \frac{c_j \cdot c_i^2}{n^2} \quad \text{and} \quad \mathbb{E}[f_{iS}] = \sum_{\mathcal{C}_j \in S} \frac{c_i \cdot c_j^2}{n^2}.$$

Let \mathcal{C}_i be a color and $\overline{\mathcal{C}}_i$ be the set of all other colors, defined as $\overline{\mathcal{C}}_i = C \setminus \mathcal{C}_i$. We observe that after one round the new number of nodes supporting \mathcal{C}_i is a random variable

$$c'_i = c_i + \sum_{j \neq i} f_{ji} - \sum_{j \neq i} f_{ij} = c_i + f_{\overline{\mathcal{C}}_i i} - f_{i \overline{\mathcal{C}}_i}.$$

Since all nodes perform their choices independently, the first sum $f_{\overline{\mathcal{C}}_i i}$ has a binomial distribution with parameters $f_{\overline{\mathcal{C}}_i i} \sim B(n - c_i, c_i^2/n^2)$. Furthermore, every node of color \mathcal{C}_i changes its color away from \mathcal{C}_i to any other opinion with probability $p_i^{\text{away}} = \sum_{j \neq i} c_j^2/n^2$. Therefore, the second sum $f_{i \overline{\mathcal{C}}_i}$ also has a binomial distribution with parameters $f_{i \overline{\mathcal{C}}_i} \sim B(c_i, p_i^{\text{away}})$. That is, we have in expectation

$$\mathbb{E}[c'_i] = c_i + \frac{(n - c_i) c_i^2}{n^2} - \frac{c_i}{n^2} \sum_{\mathcal{C}_j \neq \mathcal{C}_i} c_j^2. \quad (1)$$

Note that these expected values are monotone w.r.t. the current size. This is described more formally in the following observation.

Observation 3. Let \mathcal{C}_r and \mathcal{C}_s be two colors. It holds that $c_r \leq c_s \Rightarrow \mathbb{E}[c'_r] \leq \mathbb{E}[c'_s]$.

Proof. We first rewrite (1) as

$$\mathbb{E}[c'_i] = c_i + \frac{c_i^2}{n} - \frac{c_i}{n^2} \sum_{\mathcal{C}_j} c_j^2 = c_i \left(1 + \frac{c_i}{n} - \sum_{\mathcal{C}_j} \frac{c_j^2}{n^2} \right).$$

Using this representation of $\mathbb{E}[c'_i]$ gives us

$$\mathbb{E}[c'_r] = c_r \left(1 + \frac{c_r}{n} - \sum_{\mathcal{C}_j} \frac{c_j^2}{n^2} \right) \stackrel{c_r \leq c_s}{\leq} c_s \left(1 + \frac{c_s}{n} - \sum_{\mathcal{C}_j} \frac{c_j^2}{n^2} \right) = \mathbb{E}[c'_s]. \quad \square$$

For the following lemma, recall that $\mathcal{A} = \mathcal{C}_1$ denotes the dominant color of size $a = c_1$ and $\mathcal{B} = \mathcal{C}_2$ denotes the second largest color of size $b = c_2$.

Lemma 4. *Let \mathcal{A} be the dominating color and \mathcal{B} be the second largest color. Assume that $a - b > z \cdot \sqrt{n \log n}$. There exists a constant z such that $a' - b' > (a - b)(1 + a/4n)$ with high probability.*

In the following proof we utilize certain methods which have also been used in [17] for the two-party voting process with two choices in more general graphs.

Proof. First we observe that

$$\begin{aligned}
\mathbb{E}[a' - b'] &= a + \mathbb{E}[f_{\overline{\mathcal{A}}\mathcal{A}}] - \mathbb{E}[f_{\mathcal{A}\overline{\mathcal{A}}}] - b - \mathbb{E}[f_{\overline{\mathcal{B}}\mathcal{B}}] + \mathbb{E}[f_{\mathcal{B}\overline{\mathcal{B}}}] \\
&= a + (n - a) \cdot \frac{a^2}{n^2} - \frac{a}{n^2} \sum_{c_i \neq \mathcal{A}} c_i^2 - b - (n - b) \cdot \frac{b^2}{n^2} + \frac{b}{n^2} \sum_{c_i \neq \mathcal{B}} c_i^2 \\
&= a - b + \frac{1}{n^2} \left(a^2 n - a^3 - b^2 n + b^3 - a \sum_{c_i \neq \mathcal{A}} c_i^2 + b \sum_{c_i \neq \mathcal{B}} c_i^2 \right) \\
&= a - b + \frac{1}{n^2} \left(n(a^2 - b^2) - a \left(a^2 + \sum_{c_i \neq \mathcal{A}} c_i^2 \right) + b \left(b^2 + \sum_{c_i \neq \mathcal{B}} c_i^2 \right) \right) \\
&= a - b + \frac{1}{n} (a^2 - b^2) - \frac{1}{n^2} \left(a \sum_{c_i} c_i^2 - b \sum_{c_i} c_i^2 \right) \\
&= a - b + \frac{(a - b)(a + b)}{n} - \frac{1}{n^2} \sum_{c_i} c_i^2 (a - b) \\
&= (a - b) \cdot \left(1 + \frac{(a + b)}{n} - \frac{1}{n^2} \sum_{c_i} c_i^2 \right).
\end{aligned}$$

We now use that \mathcal{A} and \mathcal{B} are the largest and second largest colors, respectively, to bound the sum $\sum_{c_i} c_i^2$ as follows.

$$\sum_{c_i} c_i^2 = a^2 + \sum_{c_i \neq \mathcal{A}} c_i^2 \leq a^2 + \sum_{c_i \neq \mathcal{A}} c_i \cdot b = a^2 + (n - a) \cdot b \leq a^2 + n \cdot b$$

Therefore, we obtain

$$\begin{aligned}
\mathbb{E}[a' - b'] &\geq (a - b) \left(1 + \frac{(a + b)}{n} - \frac{a^2 + n \cdot b}{n^2} \right) \\
&\geq (a - b) \left(1 + \frac{a}{n} \cdot \left(1 - \frac{a}{n} \right) \right)
\end{aligned}$$

and since $a \leq n/2$ we finally get

$$\mathbb{E}[a' - b'] \geq (a - b) \left(1 + \frac{a}{2n} \right).$$

We now apply Chernoff bounds to $a' - b'$. Let $\delta_1, \delta_2, \delta_3, \delta_4$ be defined as

$$\delta_1 = \frac{2\sqrt{n \log n}}{a}, \quad \delta_2 = \frac{2n\sqrt{\log n}}{\sqrt{a \sum_{c_i \neq \mathcal{A}} c_i^2}}, \quad \delta_3 = \frac{2\sqrt{n \log n}}{b}, \quad \delta_4 = \frac{2n\sqrt{\log n}}{\sqrt{b \sum_{c_i \neq \mathcal{B}} c_i^2}}$$

for the corresponding random variables $f_{\mathcal{A}\mathcal{A}}, f_{\mathcal{A}\bar{\mathcal{A}}}, f_{\mathcal{B}\mathcal{B}}, f_{\mathcal{B}\bar{\mathcal{B}}}$ with expected values $\mu_1, \mu_2, \mu_3, \mu_4$

$$\mu_1 = (n - a) \frac{a^2}{n^2}, \quad \mu_2 = \frac{a}{n^2} \sum_{c_i \neq \mathcal{A}} c_i^2, \quad \mu_3 = (n - b) \frac{b^2}{n^2}, \quad \mu_4 = \frac{b}{n^2} \sum_{c_i \neq \mathcal{B}} c_i^2.$$

Since $a \leq n/2$ we know for the second largest color \mathcal{B} that $b \geq n/2k$. Together with $a \geq n/k \geq n^{1-\varepsilon}$ we get $0 < \delta_i < 1$ and $\delta_i^2 \cdot \mu_i = \Omega(\log n)$ for $i = 1, 2, 3, 4$. We now apply Chernoff bounds to $a' - b'$ and obtain with high probability

$$a' - b' \geq (a - b) \cdot \left(1 + \frac{a}{2n}\right) - E$$

where the error term E is bounded as follows.

$$\begin{aligned} E &= \delta_1 \cdot \mu_1 + \delta_2 \cdot \mu_2 + \delta_3 \cdot \mu_3 + \delta_4 \cdot \mu_4 \\ &= \frac{2\sqrt{n \log n}}{n^2} \left(an - a^2 + \sqrt{an \sum_{c_i \neq \mathcal{A}} c_i^2} + bn - b^2 + \sqrt{bn \sum_{c_i \neq \mathcal{B}} c_i^2} \right) \\ &\leq \frac{2\sqrt{n \log n}}{n^2} \left(\sqrt{n \sum_{c_i} c_i^2} (\sqrt{a} + \sqrt{b}) + an + bn \right) \\ &\leq \frac{2\sqrt{n \log n}}{n^2} (2an + an + bn) \\ &\leq \frac{8a\sqrt{n \log n}}{n}, \end{aligned}$$

where we used that $\sum_{c_i} c_i^2 \leq \sum_{c_i} a \cdot c_i \leq an$. From the definition of the lemma we know that $(a - b) \geq z \cdot \sqrt{n \log n}$ for some constant z . If we assume that z is large enough, e.g., $z \geq 32$, then we get with high probability

$$a' - b' \geq (a - b) \cdot \left(1 + \frac{a}{4n}\right). \quad \square$$

While [Lemma 4](#) shows that the difference between colors \mathcal{A} and \mathcal{B} does indeed increase in every round with high probability, it does not cover the remaining colors \mathcal{C}_j for $j \geq 3$. To show that also the smaller colors \mathcal{C}_j do not interfere with \mathcal{A} and thus the minimum of the difference between \mathcal{A} and any \mathcal{C}_j increases, we use the following coupling.

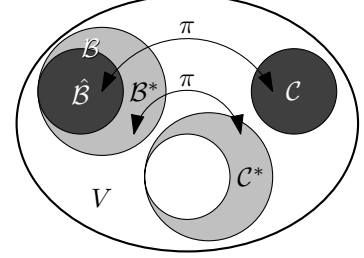
At any time step t , there exists a bijective function which maps any instance of the two-choices protocol at time t to another instance of the same protocol such that the outcome c' of the first instance is at most the outcome b' of the mapped instance.

Lemma 5. *Let \mathcal{A} be the dominating color of size a and let \mathcal{B} be the second largest color of size b . Let $\mathcal{C} \neq \mathcal{A}, \mathcal{B}$ be one of the remaining colors of size c . Furthermore, let $\pi : V \rightarrow V$ be a bijection and let P be the original process. We can couple a process $P' = P(\pi)$ to the original process P such that $c' \stackrel{(P)}{\leq} b' \stackrel{(P')}{\leq}$, where $c' \stackrel{(P)}{\leq}$ is the random variable c' in the original process and $b' \stackrel{(P')}{\leq}$ is the random variable b' in the coupled process.*

Proof. Let t be an arbitrary but fixed round. In the following, we use the notation that \mathcal{B}_t and \mathcal{C}_t are sets containing all vertices of colors \mathcal{B} and \mathcal{C} , respectively, in round t . As before, we have color sizes $b = |\mathcal{B}_t|$ and $c = |\mathcal{C}_t|$. The proof proceeds by a simple coupling argument. We start by defining $\hat{\mathcal{B}}_t, \mathcal{B}_t^*, \mathcal{C}_t^* \subseteq V$ as follows. Let $\hat{\mathcal{B}}_t$ be an arbitrary subset of \mathcal{B}_t such that $|\hat{\mathcal{B}}_t| = |\mathcal{C}_t|$. Let furthermore \mathcal{B}_t^* be defined as $\mathcal{B}_t^* = \mathcal{B}_t \setminus \hat{\mathcal{B}}_t$, and finally let \mathcal{C}_t^* be an arbitrary subset of $V \setminus (\mathcal{B}_t \cup \mathcal{C}_t)$ such that $|\mathcal{C}_t^*| = |\mathcal{B}_t^*|$.

Additionally, we construct the bijective function $\pi : V \rightarrow V$ as follows. Let $\hat{\pi}$ be an arbitrary bijection between \mathcal{C}_t and $\hat{\mathcal{B}}_t$. Let furthermore π^* be an arbitrary bijection between \mathcal{C}_t^* and \mathcal{B}_t^* . We now define π as

$$\pi(v) = \begin{cases} \hat{\pi}(v) & \text{if } v \in \mathcal{C} , \\ \hat{\pi}^{-1}(v) & \text{if } v \in \hat{\mathcal{B}} , \\ \pi^*(v) & \text{if } v \in \mathcal{C}^* , \\ \pi^{*-1}(v) & \text{if } v \in \mathcal{B}^* , \\ v & \text{if } v \in V \setminus (\mathcal{B}_t \cup \mathcal{C}_t \cup \mathcal{C}_t^*) . \end{cases}$$



It can easily be observed that π indeed forms a bijection on V . We now use π to couple a process $P' = P(\pi)$ to the original process P , to show that $b^{(P')} \geq c^{(P)}$, where the notation $b^{(P)}$ means the variable b in the original process P and $c^{(P')}$ means the variable in the coupled process P' . The coupling is now constructed such that whenever a node u samples a node $v \in V$ in the original process P , then u samples $\pi(v)$ in the coupled process P' .

Let X be the set of nodes which change their opinion to \mathcal{C} from any other color in P , that is,

$$X = \{v \in V : v \notin \mathcal{C}_t \wedge v \in \mathcal{C}_{t+1}\} .$$

Clearly, X consists of two disjoint subsets $X = \hat{X} \cup X^*$, defined as

$$\hat{X} = \{v \in V : v \notin (\mathcal{C}_t \cup \mathcal{C}_t^*) \wedge v \in \mathcal{C}_{t+1}\}$$

and

$$X^* = \{v \in V : v \in \mathcal{C}_t^* \wedge v \in \mathcal{C}_{t+1}\} .$$

The set \hat{X} consists of all nodes which change their opinion to \mathcal{C} from any other color except for nodes in \mathcal{C}^* . The set X^* contains the remaining nodes in \mathcal{C}^* which change their opinion to \mathcal{C} . Analogously to X , let Y be the set of nodes which change their opinion from \mathcal{C} to any other color in P , that is,

$$Y = \{v \in V : v \in \mathcal{C}_t \wedge v \notin \mathcal{C}_{t+1}\} .$$

Again, we have $Y = \hat{Y} \cup Y^*$ which are defined as

$$\hat{Y} = \{v \in V : v \in \mathcal{C}_t \wedge v \notin (\mathcal{C}_{t+1} \cup \mathcal{C}_{t+1}^*)\}$$

and

$$Y^* = \{v \in V : v \in \mathcal{C}_t \wedge v \in \mathcal{C}_{t+1}^*\} .$$

We now analyze the behavior of these sets in the coupled process P' . The coupling ensures the following correspondences.

Set	Process P	Process P'
X	nodes which change their color to \mathcal{C}	nodes which now belong to $\hat{\mathcal{B}}$
\hat{X}	nodes which change their color to \mathcal{C} except nodes from \mathcal{C}^*	nodes which change their color to \mathcal{B}
X^*	nodes from \mathcal{C}^* which change their color to \mathcal{C}	nodes which change their color to \mathcal{B}
Y	nodes which change their color away from \mathcal{C}	nodes which no longer belong to $\hat{\mathcal{B}}$
\hat{Y}	nodes which change their color away from \mathcal{C} but not to \mathcal{C}^*	nodes which change their color away from $\hat{\mathcal{B}}$ but not to \mathcal{B}^*
Y^*	nodes which change their color from \mathcal{C} to \mathcal{C}^*	nodes which change from $\hat{\mathcal{B}}$ to \mathcal{B}^*

Therefore, we have in P

$$c'^{(P)} = c^{(P)} + |X| - |Y|. \quad (2)$$

In P' , we first observe that $|\mathcal{B}| = |\hat{\mathcal{B}}| + |\mathcal{B}^*|$ and therefore

$$\begin{aligned} b'^{(P')} &\geq b^{(P')} + |\hat{X}| - |\hat{Y}| - (|\mathcal{B}^*| - |X^*|) \\ &\geq |\hat{\mathcal{B}}| + |\mathcal{B}^*| + |\hat{X}| - |\hat{Y}| - |\mathcal{B}^*| + |X^*| \\ &= |\hat{\mathcal{B}}| + |X| - |\hat{Y}| \\ &\geq |\hat{\mathcal{B}}| + |X| - |Y| \\ &= c^{(P)} + |X| - |Y| \end{aligned} \quad (3)$$

$$= c^{(P)} + |X| - |Y| \quad (4)$$

where the expression $|\mathcal{B}^*| - |X^*|$ in (3) is an upper bound on the number of nodes in \mathcal{B}^* which change their color away from \mathcal{B} to any other color except for $\hat{\mathcal{B}}$. Combining equations (2) and (4) gives us

$$c'^{(P)} \leq b'^{(P')}$$

which concludes the proof. \square

We now use [Lemma 4](#) and [Lemma 5](#) to show our first main result, [Theorem 1](#).

Proof. Let $\mathcal{A} = \mathcal{C}_1$ be the dominant color and $\mathcal{B} = \mathcal{C}_2$ the second largest color. Assume $a - b \geq z \cdot \sqrt{n \log n}$ for a sufficiently large constant z . From [Lemma 4](#) we know that $a' - b' \geq (a - b) \cdot (1 + a/4n)$ with high probability. Since \mathcal{B} is the second largest color, we obtain from [Lemma 5](#) for any remaining color \mathcal{C}_j with $j \geq 3$ that with high probability $a' - c'_j \geq a' - b' \geq (a - b) \cdot (1 + a/4n)$. Note that it may very well happen, especially if all colors have the same size except for \mathcal{A} , that another color \mathcal{C}_j overtakes \mathcal{B} . However, the resulting distance between \mathcal{A} and this new second largest color \mathcal{C}_j will be larger than $(a - b) \cdot (1 + a/4n)$ with high probability.

Taking Union bound over all colors, we conclude that the distance between the first color \mathcal{A} and every other color grows in every round by a factor of at least $(1 + a/4n)$ with high probability. Therefore, after $\tau = 4n/a$ rounds, the distance between \mathcal{A} and \mathcal{B} doubles with high probability. Hence, the required time for \mathcal{A} to reach size of at least $(1/2 + \varepsilon_1) \cdot n$ for a constant $\varepsilon_1 > 0$ is bounded by $O(n/a \cdot \log n)$. After additional $O(\log n)$ rounds every node has with high probability the same color \mathcal{A} , see [\[17\]](#). In each individual round, the growth described in [Lemma 4](#) takes place with high probability. Union bound over all $O(n/a \cdot \log n)$ rounds yields that the protocol indeed converges to \mathcal{A} within $O(n/a \cdot \log n)$ rounds with high probability. \square

2.1 Lower Bounds

In the previous section, we showed that the k -party voting process converges to \mathcal{A} with high probability, if the initial imbalance $a - b$ is not too small. Precisely, [Theorem 1](#) states that if $a - b \geq z \cdot \sqrt{n \log n}$ for some constant z , \mathcal{A} wins with high probability. Conversely, in the following section we examine a lower bound on the initial bias. We will show, as stated in [Theorem 7](#), that for an initial bias $a - b \leq z \cdot \sqrt{n}$ for some constant z we have a constant probability that \mathcal{B} overtakes \mathcal{A} in the first round, that is, $\Pr[a' < b'] = \Omega(1)$.

Our proof of [Theorem 7](#) is based on the normal approximation of the binomial distribution. In this context, we adapt [Theorem 2](#) and equation (6.7) from [\[27\]](#) as stated in the following theorem.

Theorem 6 (DeMoivre-Laplace limit theorem [27]). *Let X be a random variable with binomial distribution $X \sim B(N, p)$. It holds for any $x > 0$ with $x = o(N^{1/6})$ that*

$$\Pr \left[X \geq \mathbb{E}[X] + x \cdot \sqrt{\text{Var}[X]} \right] = \frac{1}{\sqrt{2\pi} \cdot x} \cdot e^{-x^2/2} \pm o(1) .$$

We now use Theorem 6 and prove Theorem 7 as follows.

Theorem 7 (Lower Bound on the Initial Bias). *For any $k \leq \sqrt{n}$ and constant z' there exists an initial assignment of colors to nodes for which $a = b + z' \cdot \sqrt{n}$ but $\Pr[a' < b'] = \Omega(1)$.*

Proof. Let $z = z'/2$ and $n' = \frac{n-k+2}{2}$. Assume that we have the following initial color distribution among the nodes.

$$(c_1, c_2, c_3, \dots, c_k) = (\lfloor n' \rfloor + \lfloor z \cdot \sqrt{n} \rfloor, \lceil n' \rceil - \lfloor z \cdot \sqrt{n} \rfloor, 1, \dots, 1) .$$

Clearly, $\sum_j c_j = n$. In the following we will omit the floor and ceiling functions for simplicity and readability reasons. First, we start by giving an upper bound on the number of nodes which change their color away from \mathcal{B} . Now recall that $f_{\mathcal{B}\overline{\mathcal{B}}}$ follows a binomial distribution $f_{\mathcal{B}\overline{\mathcal{B}}} \sim B(b, \sum_{Cj \neq \mathcal{B}} c_j^2/n^2)$ with expected value

$$\begin{aligned} \mathbb{E}[f_{\mathcal{B}\overline{\mathcal{B}}}] &= b \cdot \frac{a^2 + k - 2}{n^2} \\ &= (n' - z \cdot \sqrt{n}) \cdot \frac{(n' + z \cdot \sqrt{n})^2 + k - 2}{n^2} \\ &\leq \frac{(n' + z \cdot \sqrt{n})^3 + k - 2}{n^2} \\ &\leq \frac{n}{8} + 4z\sqrt{n} . \end{aligned}$$

Applying Chernoff bounds to $f_{\mathcal{B}\overline{\mathcal{B}}}$ gives us

$$\Pr \left[f_{\mathcal{B}\overline{\mathcal{B}}} \geq \left(1 + \sqrt{3/\mathbb{E}[f_{\mathcal{B}\overline{\mathcal{B}}}]} \right) \cdot \mathbb{E}[f_{\mathcal{B}\overline{\mathcal{B}}}] \right] \leq 1/e . \quad (5)$$

That is, with constant probability at least $1 - 1/e$ we have

$$f_{\mathcal{B}\overline{\mathcal{B}}} \leq \left(1 + \sqrt{3/\mathbb{E}[f_{\mathcal{B}\overline{\mathcal{B}}}]} \right) \cdot \mathbb{E}[f_{\mathcal{B}\overline{\mathcal{B}}}] \leq \frac{n}{8} + 4z\sqrt{n} + \sqrt{3 \cdot \mathbb{E}[f_{\mathcal{B}\overline{\mathcal{B}}}]} \leq \frac{n}{8} + (4z + 1) \cdot \sqrt{n} .$$

Secondly, we give the following lower bound on the number of nodes which change their color from \mathcal{A} to \mathcal{B} . Again, the random variable $f_{\mathcal{A}\mathcal{B}}$ for the flow from \mathcal{A} to \mathcal{B} has a binomial distribution $f_{\mathcal{A}\mathcal{B}} \sim B(a, b^2/n^2)$ with expected value

$$\mathbb{E}[f_{\mathcal{A}\mathcal{B}}] = (n' + z \cdot \sqrt{n}) \cdot \frac{(n' - z \cdot \sqrt{n})^2}{n^2} \geq \frac{(n' - z \cdot \sqrt{n})^3}{n^2} \geq \frac{(n/2 - (z + 1/2)\sqrt{n})^3}{n^2} \geq \frac{n}{8} - 4z\sqrt{n}$$

and variance

$$\text{Var}[f_{\mathcal{A}\mathcal{B}}] = \mathbb{E}[f_{\mathcal{A}\mathcal{B}}] \cdot \left(1 - \frac{(n' - z \cdot \sqrt{n})^2}{n^2} \right) \geq \frac{n}{9} \cdot \frac{1}{2} \geq \frac{n}{18} .$$

We now apply [Theorem 6](#) to $f_{\mathcal{AB}}$. Let x be a constant defined as $x = \frac{\sqrt{18}}{2}(18z + 4)$. We derive

$$\Pr \left[f_{\mathcal{AB}} \geq \mathbb{E}[f_{\mathcal{AB}}] + x \cdot \sqrt{\text{Var}[f_{\mathcal{AB}}]} \right] = \frac{1}{\sqrt{2\pi} \cdot x} e^{-x^2/2} \pm o(1) = \Omega(1) \quad .$$

That is, we have with constant probability

$$f_{\mathcal{AB}} \geq \mathbb{E}[f_{\mathcal{AB}}] + x \cdot \sqrt{\text{Var}[f_{\mathcal{AB}}]} \geq \frac{n}{8} - 4z\sqrt{n} + x \cdot \sqrt{\frac{n}{18}} \quad . \quad (6)$$

Finally, assume that in the worst case every node of colors $\mathcal{C}_3, \dots, \mathcal{C}_k$ changes to \mathcal{A} but not a single node changes away from \mathcal{A} to these colors \mathcal{C}_3 to \mathcal{C}_k . Observe that $f_{\mathcal{B}\overline{\mathcal{B}}}$ is an upper bound on $f_{\mathcal{B}\mathcal{A}}$. Therefore,

$$\begin{aligned} a' - b' &\leq (a + (k - 2) + f_{\mathcal{B}\mathcal{A}} - f_{\mathcal{B}\mathcal{A}}) - (b + f_{\mathcal{AB}} - f_{\mathcal{B}\overline{\mathcal{B}}}) \\ &\leq a - b + (k - 2) + 2f_{\mathcal{B}\overline{\mathcal{B}}} - 2f_{\mathcal{AB}} \\ &\leq 2z \cdot \sqrt{n} + (k - 2) + 2f_{\mathcal{B}\overline{\mathcal{B}}} - 2f_{\mathcal{AB}} \\ &\leq (2z + 1) \cdot \sqrt{n} + 2f_{\mathcal{B}\overline{\mathcal{B}}} - 2f_{\mathcal{AB}} \quad . \end{aligned}$$

We plug in (5) and (6) to bound the random variables $f_{\mathcal{AB}}$ and $f_{\mathcal{B}\overline{\mathcal{B}}}$ and obtain with constant probability

$$\begin{aligned} a' - b' &\leq (2z + 1) \cdot \sqrt{n} + 2 \left(\frac{n}{8} + (4z + 1)\sqrt{n} \right) - 2 \left(\frac{n}{8} - 4z\sqrt{n} + x \cdot \sqrt{\frac{n}{18}} \right) \\ &= (2z + 1 + 8z + 2 + 8z - 2x/\sqrt{18}) \cdot \sqrt{n} \\ &= (18z + 3 - 2x/\sqrt{18}) \cdot \sqrt{n} \end{aligned}$$

which gives us for $x = \frac{\sqrt{18}}{2}(18z + 4)$

$$a' - b' < 0 \quad .$$

Therefore, we have $\Pr[a' < b'] = \Omega(1)$ and thus we conclude that color \mathcal{B} prevails with constant probability. \square

Corollary 8. *There exists an initial color assignment for which $a = b + z' \cdot \sqrt{n}$ but color \mathcal{B} wins with constant probability.*

Theorem 9 (Lower Bound on the Run Time). *Assume the initial bias is exactly $z\sqrt{n \log n}$ for some constant z . The number of rounds required for the k -party voting process defined in [Algorithm 1](#) to converge is at least $\Omega(n/a + \log n)$ with constant probability.*

Proof. Let $a(t)$ denote the size of color \mathcal{A} in round t . Assume \mathcal{A} is the largest color of initial size $a(0) = n/k + z \cdot \sqrt{n \log n}$. Furthermore, assume that $k \geq 3 \cdot z$. We show by induction on the rounds that $a(t) \leq a(0) \cdot (1 + 3 \cdot a(0)/n)^t$ for $1 \leq t \leq n/10 \cdot a(0)$ with probability $1 - \frac{t}{n^2}$.

First we note that

$$a(t) \leq a(0) \cdot \left(1 + 3 \cdot \frac{a(0)}{n} \right)^t \leq a(0) \cdot \left(1 + 3 \cdot \frac{a(0)}{n} \right)^{\frac{n}{10a(0)}} \leq a(0) \cdot e^{1/2} \leq 2 \cdot a(0)$$

$$\text{and} \quad a(t) \geq a(0) \quad , \quad (7)$$

which forms a coarse bound on $a(t)$. We now prove the induction claim. The base case holds trivially. Consider step $t + 1$. By induction hypothesis we have with probability at least $1 - t/n^2$ that $a(t) \leq a(0) \cdot (1 + 3 \cdot a(0)/n)^t$. Note that we have with high probability

$$\begin{aligned} a(t+1) &\leq a(t) + f_{\mathcal{AA}} \\ &\leq a(t) + \left(1 + \frac{\sqrt{3 \log n}}{\sqrt{\mathbb{E}[f_{\mathcal{AA}]}}}\right) \cdot \mathbb{E}[f_{\mathcal{AA}}] \quad , \end{aligned}$$

where the latter inequality follows by Chernoff bounds. Using (7), we derive

$$\begin{aligned} a(t+1) &\leq a(t) + \left(1 + \frac{\sqrt{3 \log n}}{\sqrt{a(t)^2/(2 \cdot n)}}\right) \frac{a(t)^2}{n} \\ &\leq a(t) + \left(1 + \frac{\sqrt{3 \log n}}{\sqrt{a(0)^2/(2 \cdot n)}}\right) \frac{a(t)^2}{n} \\ &\leq a(t) + \frac{3}{2} \cdot \frac{a(t)^2}{n} \\ &= a(t) \cdot \left(1 + \frac{3}{2} \cdot \frac{a(t)}{n}\right) \\ &\leq a(t) \cdot \left(1 + \frac{3 \cdot a(0)}{n}\right) \quad . \end{aligned}$$

From the induction hypothesis we therefore obtain

$$\begin{aligned} a(t+1) &\leq a(0) \cdot \left(1 + \frac{3 \cdot a(0)}{n}\right)^t \cdot \left(1 + \frac{3 \cdot a(0)}{n}\right) \\ &= a(0) \cdot \left(1 + \frac{3 \cdot a(0)}{n}\right)^{t+1} \quad . \end{aligned}$$

Using a union bound to account for all errors, we derive that with probability at least $1 - (t+1)/n$ we have $a(t+1) \leq a(0) \cdot (1 + 3 \cdot a(0)/n)^{t+1}$, which completes the proof of the induction. It remains to argue that the voting time is at least $\Omega(\log n)$ with constant probability. This follows from the two-choices voting process with two colors [17]. \square

3 One Bit of Additional Memory

In this section we investigate a modified protocol where each node is allowed to store and transmit one additional bit. More precisely, each node has a bit which stores either the value `TRUE` or `FALSE`. We will say a node's bit *is set* if it has the value `TRUE`, and it *is not set* if it has the value `FALSE`. Furthermore, the modified process runs in multiple *phases* which consist of one or many rounds each. The modified voting process is formally defined in Algorithm 2. Note that in Algorithm 2 the variable ℓ is a large constant and U is an upper bound on $b/(c_1 - c_2)$. Since the process runs in multiple phases of length $\Theta(\log k + \log \log n)$ each, we assume that every node has knowledge of $\ell \cdot U$, n and k . In the following, we give a high-level description of the protocol.

```

Algorithm memory( $G = (V, E)$ ,  $\text{color} : V \rightarrow C$ ,  $\text{bit} : V \rightarrow \{\text{TRUE}, \text{FALSE}\}$ )
  for phase  $s = 1$  to  $\ell \log(U) + \log \log n$  do
    at each node  $v$  do in parallel                                     /* Round 1: two-choices */
      let  $u_1, u_2 \in N(v)$  uniformly at random;
      if  $\text{color}(u_1) = \text{color}(u_2)$  then
         $\text{color}(v) \leftarrow \text{color}(u_1)$ ;
         $\text{bit}(v) \leftarrow \text{TRUE}$ ;
      else
         $\text{bit}(v) \leftarrow \text{FALSE}$ ;
    for round  $t = 2$  to  $2 \log |C| + 2 \log \log n$  do                /* Rounds 2 to  $2 \log |C| + 2 \log \log n$  */
      at each node  $v$  do in parallel                                  /* Round  $t$ : bit-propagation */
        let  $u \in N(v)$  uniformly at random;
        if  $\text{bit}(u)$  then
           $\text{color}(v) \leftarrow \text{color}(u)$ ;
           $\text{bit}(v) \leftarrow \text{TRUE}$ ;

```

Algorithm 2: Distributed Voting Protocol with One Bit of Memory

At the beginning of each phase, we perform a step similar to the previous two-choices protocol analyzed in [Section 2](#). In this so-called *two-choices* round, each node queries two neighbors sampled uniformly at random with replacement. If these two nodes' opinions coincide, their opinion is adopted and the node sets its bit to `TRUE`, otherwise the bit is set to `FALSE`. The remaining rounds of each phase are spent propagating the bits. Note that if we assume that each node has knowledge of n/a , the run time can be further reduced to $O((\log(a/(c_1 - c_2)) + \log \log n) \cdot (\log n/a + \log \log n))$, given n/a is smaller than $k^{o(1)}$.

We start our analysis by arguing in [Lemma 10](#) that the number of bits set during the two-sample step is concentrated around the expected value. We will then use the results by Karp et al. [34] to conclude that after the bit-propagation rounds the number of set bits is n with high probability. Finally, we will prove in [Lemma 14](#) that the relative number of bits set for *large* colors remains close to the initial (relative) value during the propagation steps. Together with the growth of the total number of set bits this leads to a growth of the imbalance towards \mathcal{A} by at least a constant factor during each phase.

We will use $x(t)$ to denote the random variable for the number of nodes with set bits in a round t . Accordingly, $x(1)$ is the number of bits set after the two-choices round. Additionally, we will use $x_j(t)$ to denote the number of nodes of color \mathcal{C}_j which have their bit set in a round t . Furthermore, when analyzing the growth in $x(t)$ and $x_j(t)$, we will assume that $x(t-1)$ and $x_j(t-1)$ are fixed, respectively.

In the following lemmas we will analyze an arbitrary but fixed phase.

Lemma 10. *After the two-choices round, at least $\Omega(n/k)$ bits are set with high probability.*

Proof. The probability for one node to open connections to two nodes of the same color is $p_{\text{two-choices}} = \sum_{\mathcal{C}_j} \frac{c_j^2}{n^2}$. This probability is minimized if all colors are of the same size n/k and therefore $p_{\min} = \frac{1}{n^2} \cdot \sum_{\mathcal{C}_j} \frac{n^2}{k^2} = \frac{1}{k}$. Since all nodes open connections independently, the random variable for the number of bits set after the two-choices round, $x(1)$, has a binomial distribution

with expected value at least $\mathbb{E}[x(1)] \geq n/k$. Applying Chernoff bounds to $x(1)$ gives us

$$\Pr \left[x(1) \leq \left(1 - 2\sqrt{\frac{k \log n}{n}} \right) \frac{n}{k} \right] \leq e^{-\frac{4kn \log n}{2kn}} = n^{-2} . \quad \square$$

From the lemma above we obtain that we have at least $x(1) = n/k \cdot (1 - o(1)) = \Omega(n/k)$ bits set after the first round with high probability.

Lemma 11 (Pull Rumor Spreading [34]). *After at most $T = O(\log k + \log \log n)$ bit propagation rounds, we have $x(T) = n$ with high probability. Furthermore, $1 \leq x(t+1)/x(t) \leq 2 + o(1)$ and there exists a monotonically increasing function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that $x(t) = x(1) \cdot f(t) \cdot (1 \pm 1/n^{\Omega(1)})$ with high probability.*

In the following, we focus on the colors that are present among those nodes which have their bit set. We start by showing that the initial number of bits is well-concentrated around the expectation for colors which are *large enough*.

Lemma 12. *For any color \mathcal{C}_j with $c_j = \Omega(\sqrt{n \log n})$ the number of nodes of color \mathcal{C}_j which have their bit set after the two-choices round is concentrated around the expected value, i.e.,*

$$x_j(1) = \mathbb{E}[x_j(1)] (1 \pm O(\sqrt{\log n}/\sqrt{\mathbb{E}[x_j(1)]}))$$

with high probability. If $c_j = O(\sqrt{n \log n})$, then $x_j(1) = O(\log n)$ with high probability.

Proof. Let \mathcal{C}_j be an arbitrary but fixed color with $c_j > 3\sqrt{n \log n}$. The number of nodes of color \mathcal{C}_j which have their bit set after the two-choices round has a binomial distribution $x_j(1) \sim B(n, c_j^2/n^2)$ with expected value $\mathbb{E}[x_j(1)] = c_j^2/n > 9 \log n$. We apply Chernoff bounds to $x_j(1)$ and obtain

$$\Pr \left[|x_j(1) - \mathbb{E}[x_j(1)]| > 3\sqrt{\frac{\log n}{\mathbb{E}[x_j(1)]}} \cdot \mathbb{E}[x_j(1)] \right] \leq n^{-2} .$$

That is, we have $|x_j(1) - \mathbb{E}[x_j(1)]| \leq 3\sqrt{\log n \cdot \mathbb{E}[x_j(1)]}$ with high probability. The second statement can be shown in an analogous way. \square

Lemma 13. *Let \mathcal{C}_j be a color with at least $x_j(t) = \Omega(\log n)$ bits set in a round t . Assume $x(t)$ and $x_j(t)$ are given and they are concentrated around their mean. Then we have $\mathbb{E}[x_j(t+1)|x(t), x_j(t)] = x_j(t) + \frac{n-x(t)}{n} \cdot x_j(t)$. The number of nodes of color \mathcal{C}_j which have their bit set in round $t+1$ is with high probability concentrated around the expected value*

$$x_j(t+1) = \mathbb{E}[x_j(t+1)|x_j(t), x(t)] \left(1 \pm O \left(\frac{\sqrt{\log n}}{\sqrt{\mathbb{E}[x_j(t+1)|x(t), x_j(t)]}} \right) \right) .$$

Proof. Let p_j be the probability that v has color \mathcal{C}_j in round $t+1$, given that v has its bit set in

round $t + 1$. We have $p_j = \Pr[v \in \mathcal{C}_j(t + 1) | \text{bit}_v(t + 1) = \text{TRUE}, x_j(t), x(t)] = x_j(t)/x(t)$, since

$$\begin{aligned} \Pr[v \in \mathcal{C}_j(t + 1) | \text{bit}_v(t + 1) = \text{TRUE}, x_j(t), x(t)] &= \frac{\Pr[v \in \mathcal{C}_j(t + 1) \wedge \text{bit}_v(t + 1) = \text{TRUE} | x_j(t), x(t)]}{\Pr[\text{bit}_v(t + 1) = \text{TRUE} | x_j(t), x(t)]} \\ &= \frac{\overbrace{\frac{x_j(t)}{n} \left(\frac{n - x(t)}{n} \right)}^{(i)} + \overbrace{\frac{x_j(t)}{n}}^{(ii)}}{\underbrace{\frac{x(t)}{n}}_{(iii)} + \underbrace{\left(1 - \frac{x(t)}{n} \right) \frac{x(t)}{n}}_{(iv)}} = \frac{x_j(t)}{x(t)} \cdot \frac{1 - \frac{x(t)}{n} + 1}{1 + 1 - \frac{x(t)}{n}}. \end{aligned}$$

Observe that in the above equation the probability for a node to have color \mathcal{C}_j and the bit set at time $t + 1$ is as follows.

- (i) is the probability that a node has color \mathcal{C}_j and the bit set at time t and selects a node without a bit set
- (ii) a node chooses another node which has color \mathcal{C}_j and the bit set
- (iii) the probability for choosing a node with a set bit
- (iv) the probability for choosing a node which selects another node without the bit set

Consequently, the number of nodes which have color \mathcal{C}_j in the next round has expected value $\mu = \mathbb{E}[x_j(t + 1) | x(t + 1), x_j(t), x(t)] = x_j(t) \cdot x(t + 1)/x(t)$. We apply Chernoff bounds to $x_j(t + 1)$ and obtain

$$\Pr \left[|x_j(t + 1) - \mu| > 3\sqrt{\frac{\log n}{\mu}} \cdot \mu \mid x_j(t), x(t), x(t + 1) \right] \leq n^{-2}.$$

Assuming $x(t)$ fulfills [Lemma 10](#), we have $x(t + 1) = \mathbb{E}[x(t + 1) | x(t)] \cdot (1 \pm O(\sqrt{k \log n}/\sqrt{n}))$ [\[34\]](#), we obtain the lemma. \square

Lemma 14. *Let \mathcal{A} be the dominant color of size a and \mathcal{B} the second largest color of size b . Let a' and b' be the number of nodes of colors \mathcal{A} and \mathcal{B} , respectively, after the bit-propagation phase. Let $T = 2(\log k + \log \log n)$. Given $x(1)$ and assuming it is concentrated around the expected value, we have with high probability after T bit-propagation rounds*

$$a' \geq \frac{a^2}{x(1)} \cdot \left(1 - O\left(\frac{T \cdot \sqrt{n \log n}}{a}\right) \right) \quad \text{and} \quad b' \leq \frac{b^2}{x(1)} \cdot \left(1 + O\left(\frac{T \cdot \sqrt{n \log n}}{b}\right) \right) + \log^2 n.$$

Furthermore, for any other color \mathcal{C}_j of size c_j it holds with high probability that

$$c'_j \leq \frac{c_j^2}{x(1)} \cdot \left(1 + O\left(\frac{T \cdot \sqrt{n \log n}}{c_j}\right) \right) + k^2 \cdot \log^4 n.$$

Proof. Let $a_i = x_1(i)$ be a sequence of random variables for the number of nodes of color \mathcal{A} which have their bit set in round i . In the following proof, whenever we condition on a_j or $x(j)$ for any j , we assume that they are concentrated around their mean according to [Lemma 11](#), [Lemma 12](#), and [Lemma 13](#).

According to [Lemma 13](#) we know that

$$\mathbb{E}[a_{i+1}|a_i, x(i+1), x(i)] = \frac{x(i+1)}{x(i)} \cdot a_i .$$

Note that $\mathbb{E}[a_{i+1}|a_i] \geq a_i$. Therefore we have

$$\Pr \left[a_{i+1} < \frac{x(i+1)}{x(i)} \cdot a_i \cdot \left(1 - \frac{3\sqrt{\log n}}{\sqrt{a_i}} \right) \middle| a_i, x(i-1), x(i) \right] \leq n^{-2} .$$

The total number of bits set in the round $i+1$, given the total number of bits in round i , is independent of the color distribution among these nodes in round i , that is, for any $\beta \leq \gamma$ it holds for any α that

$$\Pr[x(i+1) = \alpha | x_j(i) = \beta, x(i) = \gamma] = \Pr[x(i+1) = \alpha | x(i) = \gamma] .$$

We therefore have for any $\tau > i$

$$\Pr \left[a_{i+1} < \frac{x(i+1)}{x(i)} \cdot a_i \cdot \left(1 - \frac{3\sqrt{\log n}}{\sqrt{a_i}} \right) \middle| a_i, x(1), \dots, x(\tau) \right] \leq n^{-2} .$$

The equation above means that the distribution of the colors among the nodes with the bit set at time $i+1$, given $x(1) \dots x(i+1)$, is independent of the number of nodes with the bit set at times $i+2, \dots, \tau$.

Recall that, given a_1 , $a_i = \Omega(a_1)$ with high probability and therefore we have for given a_1 , a_i , $x(i-1)$, $x(i)$, and a constant ζ with high probability

$$a_{i+1} \geq \frac{x(i+1)}{x(i)} \cdot a_i \cdot \left(1 - \zeta \cdot \frac{\sqrt{\log n}}{\sqrt{a_1}} \right) . \quad (8)$$

Define $T = O(\log n/a + \log \log n)$ such that $x(T) = n$ with high probability according to [\[34\]](#). We now show by induction that, given a_1 , $x(1), \dots, x(T)$, and a constant ζ ,

$$a_T \geq \frac{x(T)}{x(1)} \cdot a_1 \cdot \left(1 - \zeta \cdot \frac{\sqrt{\log n}}{\sqrt{a_1}} \right)^T \quad (9)$$

with high probability. The base case for round $t = 1$ obviously holds. For the step from t to $t+1$ we use (8) as follows.

$$\begin{aligned} a_{t+1} &\stackrel{(8)}{\geq} \frac{x(t+1)}{x(t)} \cdot a_t \cdot \left(1 - \zeta \cdot \frac{\sqrt{\log n}}{\sqrt{a_1}} \right) \\ &\stackrel{\text{IH}}{\geq} \frac{x(t+1)}{x(t)} \cdot \frac{x(t)}{x(1)} \cdot a_1 \cdot \left(1 - \zeta \cdot \frac{\sqrt{\log n}}{\sqrt{a_1}} \right)^t \cdot \left(1 - \zeta \cdot \frac{\sqrt{\log n}}{\sqrt{a_1}} \right) \\ &\geq \frac{x(t+1)}{x(1)} \cdot a_1 \cdot \left(1 - \zeta \cdot \frac{\sqrt{\log n}}{\sqrt{a_1}} \right)^{t+1} \end{aligned}$$

This concludes the induction. We apply the Bernoulli inequality to (9) and obtain

$$a_T \geq \frac{x(T)}{x(1)} \cdot a_1 \cdot \left(1 - \zeta \cdot \frac{T \cdot \sqrt{\log n}}{\sqrt{a_1}} \right) .$$

A similar upper bound can be computed for any *large* color. Let $\mathcal{B} = \mathcal{C}_2$ be the second largest color. For $b_i = x_2(i)$ we obtain with high probability

$$b_T \leq \frac{x(T)}{x(1)} \cdot b_1 \cdot \left(1 + \zeta \cdot \frac{T \cdot \sqrt{\log n}}{\sqrt{b_1}}\right).$$

In the following we analyze how the gap between \mathcal{A} and \mathcal{B} changes during one phase. We use the result from [Lemma 12](#) for a_1 in (3) and obtain

$$\begin{aligned} a' &\geq \frac{n}{x(1)} \cdot a_1 \cdot \left(1 - \zeta \cdot \frac{T \cdot \sqrt{\log n}}{\sqrt{a_1}}\right) \\ &\geq \frac{n}{x(1)} \cdot \frac{a^2}{n} \cdot \underbrace{\left(1 - \zeta \cdot \frac{T \cdot \sqrt{\log n}}{\sqrt{a_1}}\right)}_{(I)} \cdot \underbrace{\left(1 - \frac{3\sqrt{\log n} \cdot \sqrt{n}}{a}\right)}_{(II)}, \end{aligned}$$

where the second expression in parentheses, (II), is asymptotically dominated by the first one, (I). Therefore, there is a ζ' such that

$$a' \geq \frac{a^2}{x(1)} \cdot \left(1 - \zeta' \cdot \frac{T \cdot \sqrt{\log n}}{\sqrt{a_1}}\right). \quad (10)$$

As before, we can apply a similar calculation for the upper bound of any color \mathcal{B} as long as b is large enough, i.e., $b \geq \sqrt{n \log n}$. We therefore obtain with high probability

$$b' \leq \frac{b^2}{x(1)} \cdot \left(1 + \zeta' \cdot \frac{T \cdot \sqrt{\log n}}{\sqrt{b_1}}\right). \quad (11)$$

Finally, the same calculation can be also applied to any other color \mathcal{C}_j of size c_j . However, if c_j is between $\sqrt{n}/\log n$ and $\sqrt{n} \cdot \log n$, then we observe that after the two-choices round we have at most $O(\log^2 n)$ bits set for \mathcal{C}_j . Since in any step the color can increase by at most a factor of $2 \cdot (1 + o(1/\log^2 n))$ with high probability, we have in the end at most $O(k^2 \cdot \log^4 n)$ nodes of color \mathcal{C}_j . Since $k \leq n^\epsilon$, in the next two-choices phase this color will disappear with high probability.

Taking all contributions into consideration, we observe that there always exists a constant ζ' such that (10) and (11) are satisfied. \square

We are now ready to put all pieces together and prove our main theorem, [Theorem 2](#), which is restated as follows.

Theorem 2. Let $G = K_n$ be the complete graph with n nodes. The memory voting process defined in [Algorithm 2](#) on G converges within $O((\log(c_1/(c_1 - c_2)) + \log \log n) \cdot (\log k + \log \log n))$ time steps to \mathcal{A} , with high probability, if $c_1 - c_2 \geq z \cdot \sqrt{n \log^3 n}$ for some constant z .

Proof. Assume $x(1)$ is given and concentrated around its expected value. In the following proof, we assume that $b \geq \sqrt{n \log n}$. Recall that in the statement of [Theorem 2](#) we assume $a - b \geq z \cdot \sqrt{n \log^3 n}$. Let $T = 2(\log k + \log \log n)$. From the bounds on a' and b' from [Lemma 14](#)

we obtain the following inequality which holds with high probability.

$$\begin{aligned}
a' - b' &\geq \frac{a^2 - b^2}{x(1)} - \frac{\zeta \cdot T \cdot \sqrt{\log n}}{x(1)} \cdot \left(\frac{a^2}{\sqrt{a_1}} + \frac{b^2}{\sqrt{b_1}} \right) \\
&\geq \frac{a^2 - b^2}{x(1)} - \frac{2 \cdot \zeta \cdot T \cdot \sqrt{\log n}}{x(1)} \cdot \frac{a^2}{\sqrt{a_1}} \\
&\geq \frac{a - b}{x(1)} \cdot \left((a + b) - \frac{2 \cdot \zeta \cdot T \cdot \sqrt{\log n}}{a - b} \cdot \frac{a^2}{\sqrt{a_1}} \right)
\end{aligned}$$

(using $a_1 = a^2/n \cdot (1 \pm o(1))$ with high probability according to [Lemma 12](#))

$$\geq \frac{a - b}{x(1)} \cdot \left((a + b) - \frac{2 \cdot \zeta \cdot T \cdot \sqrt{\log n} \cdot a^2 \cdot \sqrt{n}}{(a - b) \cdot a \cdot (1 - o(1))} \right)$$

(using $a - b \geq z \cdot \sqrt{n \log^3 n}$)

$$\geq \frac{a - b}{x(1)} \cdot \left((a + b) - \frac{2 \cdot \zeta \cdot a}{z \cdot (1 - o(1))} \right)$$

Now if z is large enough, we obtain for a small positive constant $\varepsilon = \varepsilon(z)$ that

$$a' - b' \geq (a - b) \cdot \left(\frac{a(1 - \varepsilon) + b}{x(1)} \right). \quad (12)$$

Observe that $\mathbb{E}[x(1)] = \sum_{c_j} c_j^2/n$. Since $c_j \leq a$, it holds that $\mathbb{E}[x(1)] \leq 1/n \sum_{c_j} c_j \cdot a = a$. Applying Chernoff bounds to $x(1)$ yields $x(1) \leq a(1 + o(1))$ with high probability. We use this in (12) and obtain with high probability

$$a' - b' \geq (a - b) \cdot \left(1 - \varepsilon + \frac{b}{a} \right).$$

We now distinguish the following two cases.

Case 1: $b \geq 2\varepsilon \cdot a$. For z large enough and thus ε small enough the gap between \mathcal{A} and \mathcal{B} grows by a constant factor of at least $1 + \varepsilon$.

Case 2: $b < 2\varepsilon \cdot a$. We consider the ratio between a' and b' based on the results in (10) and (11) as follows.

$$\frac{a'}{b'} \geq \frac{\frac{a^2}{x(1)} \cdot \left(1 - \zeta \cdot \frac{\log^{\frac{3}{2}} n}{\sqrt{a_1}} \right)}{\frac{b^2}{x(1)} \cdot \left(1 + \zeta \cdot \frac{\log^{\frac{3}{2}} n}{\sqrt{b_1}} \right)} = \frac{a^2}{b^2} \cdot \frac{1 - o(1)}{1 + o(1)} \geq \frac{a^2}{b^2} \cdot (1 - o(1)) \quad (13)$$

Case 1 implies that we initially have $b = \Theta(c_1)$ and within $O(\log(b/(c_1 - c_2)))$ phases we have $b \leq 2\varepsilon \cdot a$. Applying further $O(\log \log n)$ phases, every color except for \mathcal{A} drops below $\sqrt{n \log n}$. As described in the proof of [Lemma 14](#), all other colors disappear in the next two-choices phase with high probability. \square

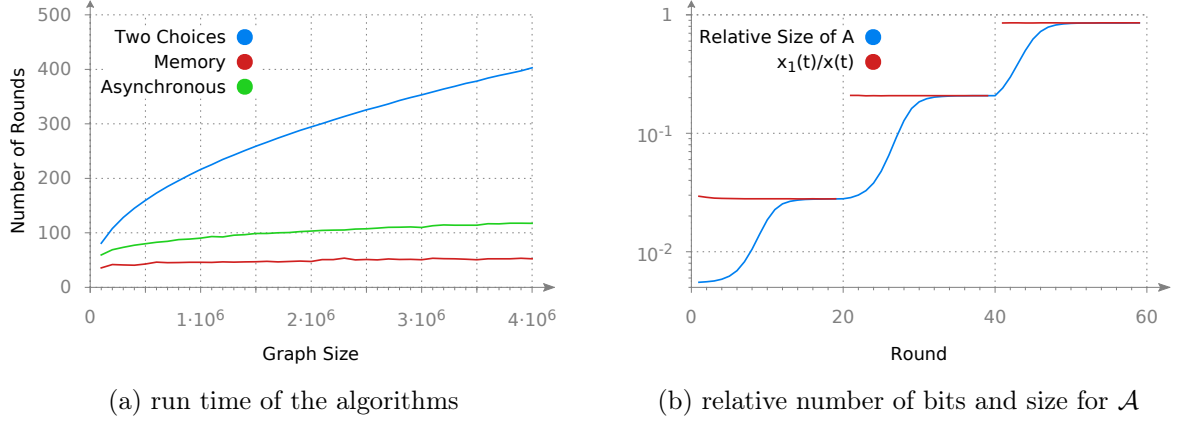


Figure 1: The left plot shows the run times for the four algorithms defined in [Algorithm 1](#), [Algorithm 2](#), and [Algorithm 3](#) for over the graph sizes. The right plot shows the relative number of bits and the relative size for \mathcal{A} for the complete graph with $n = 10^6$.

4 Simulation Results

In this section we present various simulation results to support our theoretical findings. We implemented our simulation to run on a shared memory machine and simulate the distributed system.

First, we measured the run time of our simulation until the simulation converges to \mathcal{A} . We set $k \approx \sqrt{n}$, $a - b \approx \sqrt{n \log n}$, and $c_2, \dots, c_k \approx n/k$, to simulate the processes for varying n . Our simulation results indicate that, as shown in [Theorem 2](#), the memory based k-party voting protocol outperforms the classical two-choices approach by orders of magnitude, see [Figure 1a](#).

Secondly, we investigated the behavior of the bits as used in the analysis in [Section 3](#). We therefore plotted in [Figure 1b](#) the relative number of nodes which have a bit for color \mathcal{A} among other nodes which have a bit. That is, our plots show $x_1(t)/x(t)$ for every round t . Additionally, the relative number of nodes of color \mathcal{A} is shown in the plot. The simulation indeed confirms an exponential growth of \mathcal{A} during the bit propagation phase.

Finally, an asynchronous variant of the voting process described in [Algorithm 3](#) has been simulated. In this asynchronous setting, each node is activated by a random clock which ticks according to a Poisson distribution. In expectation, each node ticks once per time unit. For our simulation, we ran a sequentialized process where at every loop iteration a node was chosen uniformly at random to perform its operation. We conclude that our simulation results empirically show that the asynchronous memory-based voting process performs very well for practical application.

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```

Algorithm asynchronous( $G = (V, E)$ ;  $\text{color} : V \rightarrow C$ ;  $\text{bit} : V \rightarrow \{\text{TRUE}, \text{FALSE}\}$ )
  at each node  $v$  do asynchronously
    for  $\text{phase } s = 1$  to  $10 \cdot \log_2 |V|$  do
      let  $u_1, u_2 \in N(v)$  uniformly at random;
      if  $\text{color}(u_1) = \text{color}(u_2)$  then
         $\text{color}(v) \leftarrow \text{color}(u_1)$ ;
         $\text{bit}(v) \leftarrow \text{TRUE}$ ;
      else
         $\text{bit}(v) \leftarrow \text{FALSE}$ ;
      for 2 ticks do                                     /* bit-propagation subphase */
        let  $u \in N(v)$  uniformly at random;
        if  $\text{bit}(u) = \text{TRUE}$  then
           $\text{color}(v) \leftarrow \text{color}(u)$ ;
           $\text{bit}(v) \leftarrow \text{TRUE}$ ;

```

Algorithm 3: Simulated Asynchronous Distributed Voting Protocol

5 Conclusion

We analyze the two-choices protocol as well as a simple modification of this process on the clique for $k = O(n^\epsilon)$ initial opinions. For the standard protocol, we obtain almost tight bounds on the initial bias required to ensure that the dominant color \mathcal{A} wins, as well as a bound of $O(k \cdot \log n)$ on the convergence time. The modified process builds upon the standard protocol, and uses one additional bit to be transmitted in (almost) every round. Surprisingly, this additional bit suffices to reduce the run time by an order of magnitude for plenty of initial configurations (e.g., when k is polynomial in n). More precisely, the run time is bounded by $O(\log^2 n)$ even if the initial bias is only $O(\sqrt{n \log^3 n})$. Furthermore, when $c_1 \geq (1 + 1/\log n)c_2$ and $k = O(\text{poly}(\log n))$, then the run time becomes $\text{poly}(\log \log n)$. In contrast to existing work considering $k > 2$, our algorithm ensures that the dominant color \mathcal{A} wins within a polylogarithmic number of rounds even if the bias is only $O(\sqrt{n \log^3 n})$.

References

- [1] M. A. Abdullah and M. Draief. Global majority consensus by local majority polling on graphs of a given degree sequence. *Discrete Applied Mathematics*, 180:1–10, 2015.
- [2] D. Aldous and J. Fill. Reversible Markov Chains and Random Walks on Graphs, 2002. Unpublished. <http://www.stat.berkeley.edu/~aldous/RWG/book.html>.
- [3] D. Alistarh, R. Gelashvili, and M. Vojnovic. Fast and Exact Majority in Population Protocols. In *Proceedings of the 2015 ACM Symposium on Principles of Distributed Computing, (PODC)*, pages 47–56, 2015.
- [4] D. Angluin, J. Aspnes, and D. Eisenstat. A simple population protocol for fast robust approximate majority. *Distributed Computing*, 21(2):87–102, 2008.
- [5] D. Angluin, J. Aspnes, D. Eisenstat, and E. Ruppert. The computational power of population protocols. *Distributed Computing*, 20(4):279–304, 2007.
- [6] J. Aspnes and E. Ruppert. An Introduction to Population Protocols. *Bulletin of the EATCS*, 93:98–117, 2007.
- [7] V. Auletta, I. Caragiannis, D. Ferraioli, C. Galdi, and G. Persiano. Minority Becomes Majority in Social Networks. In *Proc. WINE '15*, pages 74–88, 2015.
- [8] L. Becchetti, A. Clementi, E. Natale, F. Pasquale, and R. Silvestri. Plurality Consensus in the Gossip Model. In *Proc. SODA '15*, pages 371–390, 2015.
- [9] L. Becchetti, A. Clementi, E. Natale, F. Pasquale, R. Silvestri, and L. Trevisan. Simple Dynamics for Plurality Consensus. In *Proc. SPAA '14*, pages 247–256, 2014.
- [10] L. Becchetti, A. E. F. Clementi, E. Natale, F. Pasquale, and L. Trevisan. Stabilizing Consensus with Many Opinions. In *Proceedings of the Twenty-Sixth Annual ACM-SIAM Symposium on Discrete Algorithms (SODA)*. SIAM, 2016.
- [11] P. Berenbrink, T. Friedetzky, P. Kling, F. Mallmann-Trenn, and C. Wastell. Plurality Consensus via Shuffling: Lessons Learned from Load Balancing. *CoRR*, abs/1602.01342, 2016.
- [12] E. Berger. Dynamic Monopolies of Constant Size. *Journal of Combinatorial Theory, Series B*, 83(2):191–200, 2001.
- [13] S. Brahma, S. Macharla, S. Pal, and S. Singh. Fair Leader Election by Randomized Voting. In *Proc. ICDCIT '04*, pages 22–31, 2004.
- [14] L. Cardelli and A. Csikász-Nagy. The cell cycle switch computes approximate majority. *Scientific reports*, 2, 2012.
- [15] L. Cardelli and A. Csikász-Nagy. The Cell Cycle Switch Computes Approximate Majority. *Scientific Reports*, 2(656), 2012.
- [16] C. Cooper, R. Elsässer, H. Ono, and T. Radzik. Coalescing Random Walks and Voting on Connected Graphs. *SIAM Journal on Discrete Mathematics*, 27(4):1748–1758, 2013.

- [17] C. Cooper, R. Elsässer, and T. Radzik. The Power of Two Choices in Distributed Voting. In *Proc. ICALP '14*, pages 435–446, 2014.
- [18] C. Cooper, R. Elsässer, T. Radzik, N. Rivera, and T. Shiraga. Fast Consensus for Voting on General Expander Graphs. In *Proc. DISC '15*, pages 248–262, 2015.
- [19] C. Cooper, A. Frieze, and T. Radzik. Multiple Random Walks in Random Regular Graphs. *SIAM Journal on Discrete Mathematics*, 23(4):1738–1761, 2009.
- [20] G. Cordasco and L. Gargano. Community Detection via Semi-Synchronous Label Propagation Algorithms. In *Proc. BASNA '10*, pages 1–8, 2010.
- [21] J. Cruise and A. Ganesh. Probabilistic consensus via polling and majority rules. *Queueing Systems*, 78(2):99–120, 2014.
- [22] X. Deng and C. Papadimitriou. On the Complexity of Cooperative Solution Concepts. *Mathematics of Operations Research*, 19(2):257–266, 1994.
- [23] B. Doerr, L. A. Goldberg, L. Minder, T. Sauerwald, and C. Scheideler. Stabilizing consensus with the power of two choices. In *Proceedings of the 23rd Annual ACM Symposium on Parallelism in Algorithms and Architectures, (SPAA)*, pages 149–158, 2011.
- [24] P. Donnelly and D. Welsh. Finite particle systems and infection models. *Mathematical Proceedings of the Cambridge Philosophical Society*, 94(1):167–182, 1983.
- [25] D. Doty. Timing in Chemical Reaction Networks. In *Proc. SODA '14*, pages 772–784, 2014.
- [26] M. Draief and M. Vojnović. Convergence speed of binary interval consensus. *SIAM Journal on Control and Optimisation*, 50(3):1087–1109, 2012.
- [27] W. Feller. *An Introduction to Probability Theory and Its Applications*. Wiley, 3rd edition, 1968.
- [28] S. Frischknecht, B. Keller, and R. Wattenhofer. Convergence in (Social) Influence Networks. In *Proc. DISC '13*, pages 433–446, 2013.
- [29] D. Gifford. Weighted Voting for Replicated Data. In *Proc. SOSP '79*, pages 150–162, 1979.
- [30] E. Goles and S. Martínez. *Neural and Automata Networks*. Kluwer, 1990.
- [31] Y. Hassin and D. Peleg. Distributed Probabilistic Polling and Applications to Proportionate Agreement. *Information and Computation*, 171(2):248–268, 2001.
- [32] R. Holley and T. Liggett. Ergodic Theorems for Weakly Interacting Infinite Systems and the Voter Model. *The Annals of Probability*, 3(4):643–663, 1975.
- [33] B. W. Johnson, editor. *Design & Analysis of Fault Tolerant Digital Systems*. Addison-Wesley, 1989.
- [34] R. Karp, C. Schindelhauer, S. Shenker, and B. Vocking. Randomized Rumor Spreading. In *Proc. FOCS '00*, pages 565–574, 2000.

- [35] D. Kempe, A. Dobra, and J. Gehrke. Gossip-Based Computation of Aggregate Information. In *Proc. FOCS '03*, pages 482–491, 2003.
- [36] K. Kothapalli, S. Pemmaraju, and V. Sardeshmukh. On the Analysis of a Label Propagation Algorithm for Community Detection. In *Proc. ICDCN '13*, pages 255–269, 2013.
- [37] N. Lanchier and C. Neuhauser. Voter model and biased voter model in heterogeneous environments. *Journal of Applied Probability*, 44(3):770–787, 2007.
- [38] T. Liggett. *Interacting particle systems*. Springer Science & Business Media, 2012.
- [39] Y. Lv and T. Moscibroda. Local Information in Influence Networks. In *Proc. DISC '15*, pages 292–308, 2015.
- [40] F. Mallmann-Trenn. Bounds on the voting time in terms of the conductance. Master’s thesis, Simon Fraser University, 2014. Master’s thesis. <http://summit.sfu.ca/item/14502>.
- [41] G. B. Mertzios, S. E. Nikolettseas, C. Raptopoulos, and P. G. Spirakis. Determining Majority in Networks with Local Interactions and Very Small Local Memory. In *Automata, Languages, and Programming - 41st International Colloquium, (ICALP)*, pages 871–882, 2014.
- [42] M. Mossel and O. Tamuz. Opinion Exchange Dynamics. *CoRR*, abs/1401.4770, 2014.
- [43] T. Nakata, H. Imahayashi, and M. Yamashita. Probabilistic Local Majority Voting for the Agreement Problem on Finite Graphs. In *Proc. COCOON '99*, pages 330–338, 1999.
- [44] D. Peleg. Local majorities, coalitions and monopolies in graphs: a review. *Theoretical Computer Science*, 282(2):231–257, 2002.
- [45] D. Peleg. Immunity against Local Influence. In *Language, Culture, Computation. Computing - Theory and Technology*, volume 8001 of *LNCS*, pages 168–179. Springer, 2014.
- [46] E. Perron, D. Vasudevan, and M. Vojnović. Using Three States for Binary Consensus on Complete Graphs. In *Proc. INFOCOM '09*, pages 2527–2535, 2009.
- [47] U. Raghavan, R. Albert, and S. Kumara. Near linear time algorithm to detect community structures in large-scale networks. *Physical Review E*, 76(3):036106, 2007.